Advanced Probability - Final 17/11/18

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Attempt all questions. The total marks is 54 but the maximum you can score is 50
Time: 3 hours

If you are using a specific theorem mention it and check all the conditions.

All random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$.

1. Let $\xi_1, \xi_2 \cdots$ be independent random variables such that $E\xi_i = 0$ and $\text{Var}(\xi_i) = \sigma_i^2 < \infty$. Let $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Show that $S_n^2 - s_n^2$ is a martingale with respect to $\mathcal{F}_n = \sigma\{\xi_1, \xi_2 \cdots \xi_n\}$. **[4 marks]**

2. Show that if X_n and Y_n are submartingales with respect to filtration $\mathcal{F}_n$ then $\max(X_n, Y_n)$ is also. **[4 marks]**

3. (a) Let c be a constant. For a sequence of random variables X_n show that $X_n \xrightarrow{P} c$ if and only if $X_n \Rightarrow c$. (Hint: you might need to use $|e^{ia} - e^{ib}| \le |a - b|$) **[4 marks]**
   (b) If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$ for some constant c then show $X_n + Y_n \Rightarrow X + c$. **[4 marks]**

4. Suppose $X_1, \cdots$ are independent random variables with $P(X_m = m) = P(X_m = -m) = m^{-2}/2$, and for $m \ge 2$
$$P(X_m = 1) = P(X_m = -1) = (1 - m^{-2})/2.$$
   Let $S_n = X_1 + X_2 + \cdots + X_n$.
   (a) Let $Y_i = \text{sign}(X_i)$ where $\text{sign}(X_i) = +1$ or -1 depending on whether X_i is positive or not. Let $R_n = Y_1 + Y_2 + \cdots + Y_n$. Show that $R_n/\sqrt{n} \Rightarrow N(0,1)$. **[3 marks]**
   (b) Show that $S_n/\sqrt{n} \Rightarrow N(0,1)$. **[4 marks]**

5. Let X_n be any sequence of real valued random variables. Show that there are constants $c_n \to \infty$ so that $X_n/c_n \to 0$ a.s. **[4 marks]**

6. Let $\psi(x) = x^2$ when $|x| \le 1$ and $= |x|$ when $|x| \ge 1$. Show that if $X_1, X_2 \cdots$ are independent with $EX_n = 0$ and $\sum_n E\psi(X_n) < \infty$ then $\sum_n X_n$ converges a.s. (Hint: Use Kolmogorov's three series theorem) **[4 marks]**

7. Let X_n, $n \ge 0$ be a martingale and let $\xi_n = X_n - X_{n-1}$, $n \ge 1$ be the martingale differences. Suppose $E[X_0^2] < \infty$ and $\sum_{m=1}^n E[\xi_m^2] < \infty$ then $X_n \to X_\infty$ a.s. and in L^2. **[4 marks]**

8. Let $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ where ξ_i are i.i.d. random variables such that $|\xi_i| \le M$ a.s.. Assume that ξ is not a constant and $Ee^{\theta\xi_1} = 1$ for some $\theta < 0$. Fix $a < 0 < b$ and let $T = \inf\{n : S_n \notin (a,b)\}$.
   (a) Show that $X_n = \exp(\theta S_n)$ is a martingale with respect to $\mathcal{F}_n = \sigma(\xi_1, \xi_2 \cdots \xi_n)$. **[3 marks]**
   (b) Show that $P(T < \infty) = 1$. **[4 marks]**
   (c) Show that $P(S_T \le a) \le e^{-\theta a}$. **[5 marks]**

9. Consider a random variable X on a probability space $(\Omega, \mathcal{F}, P)$ such that $E[X^2] < \infty$.
   (a) Show that $E|X| < \infty$. **[1 marks]**
   (b) Given a sub σ-algebra $\mathcal{G} \subset \mathcal{F}$, show that $Z = E(X|\mathcal{G})$ satisfies $E[Z^2] < \infty$. **[2 marks]**
   (c) Show that $E(X|\mathcal{G})$ minimizes $E[(X - Y)^2]$ for Y such that $E[Y^2] < \infty$. **[4 marks]**